Open and other kinds of extensions over local compactifications

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Abstract

Generalizing de Vries Compactification Theorem [3] and strengthening Leader Local Compactification Theorem [14], we describe the partially ordered set $(\mathcal{L}(X), \leq)$ of all (up to equivalence) locally compact Hausdorff extensions of a Tychonoff space X. Using this description, we find the necessary and sufficient conditions which has to satisfy a map between two Tychonoff spaces in order to have some kind of extension over arbitrary given in advance Hausdorff local compactifications of these spaces; we regard the following kinds of extensions: open, quasi-open, skeletal, perfect, injective, surjective. In this way we generalize some results of V. Z. Poljakov [18].

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Introduction

In 1959, V. I. Ponomarev [19] proved that if $f : X \longrightarrow Y$ is a perfect open surjection between two normal Hausdorff spaces X and Y then its extension $\beta f : \beta X \longrightarrow \beta Y$ over Stone-Čech compactifications of these spaces is an open map; he obtained as well a more general variant of this theorem which concerns multi-valued mappings. He posed the following problem: characterize those continuous maps $f : X \longrightarrow Y$ between two Tychonoff spaces for which the map βf is open. In 1960, A. D. Taĭmanov [1] improved Ponomarev's theorem cited above by replacing "perfect" with

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"closed" (and A. V. Arhangel'skiĭ [1] generalized Taĭmanov's result for multi-valued mappings). Later on, V. Z. Poljakov [18] described the maps $f : X \longrightarrow Y$ between two Tychonoff spaces which have an *open* extension over arbitrary given in advance Hausdorff compactifications (cX, c_X) and (cY, c_Y) of X and Y respectively. His work is based on the famous Smirnov Compactification Theorem [21] which affirms that there exists an isomorphism between the partially ordered sets (= posets) $(\mathcal{K}(X), \leq)$ of all, up to equivalence, Hausdorff compactifications of a Tychonoff space X, and $(\mathcal{P}(X), \preceq)$ of all Efremovič proximities on the space X; with the help of this theorem, Ju. M. Smirnov [21] describes the maps between two Tychonoff spaces which can be extended *continuously* over arbitrary given in advance compactifications of these spaces. Analogous assertions for the Hausdorff local compactifications (= locally compact extensions) of Tychonoff spaces were proved by S. Leader [14].

In this paper we generalize Poljakov's and Leader's theorems and obtain some other results of this type. We regard the following kinds of extensions over Hausdorff local compactifications: open, quasi-open (in the sense of [15]), perfect, skeletal (in the sense of [16]), injective, surjective. We characterize the functions between Tychonoff spaces which have extensions of the kinds listed above over arbitrary given in advance local compactifications (see Theorem 3.5); in particular, in Corollary 3.7, we formulate correctly the Poljakov's answer to Ponomarev's question (V. Z. Poljakov [18] derives it from the general theorem proved by him but gives in fact only a sufficient condition (see Remark 3.8)). The characterizations of all these maps are obtained here with the help of a strengthening of the Leader Local Compactification Theorem [14] (see Theorems 2.2 and 3.1 below). We give a de Vries-type formulation of the Leader's theorem (i.e. we describe axiomatically the restrictions of the Leader's *local proximities* on the Boolean algebra $RC(X)$ of all regular closed subsets of a Tychonoff space X) and prove this new assertion independently of the Leader's theorem using only our generalization (see [5]) of de Vries Duality Theorem [3]. This permits us to use our recent general results obtained in [4, 6]. Finally, on the base of our variant of Leader's Theorem, we characterize in the language of *local contact algebras* only (i.e. without mentioning the points of the space) the poset $(\mathcal{L}(X), \leq)$ of all, up to equivalence, Hausdorff local compactifications of X, where X is a locally compact Hausdorff space (see Theorem 2.11); the algebras which correspond to the Alexandroff (one-point) compactification and to the Stone-Čech compactification of a locally compact Hausdorff space are described explicitly (see Theorem 2.12). Let us mention as well that in [8] we described, using the language of non-symmetric proximities, the surjective continuous maps which have a perfect extension over arbitrary given in advance Hausdorff local compactifications.

We now fix the notations.

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if f is a morphism of $\mathcal{C}$ with domain X and codomain Y. By $Id_{\mathcal{C}}$ we denote the identity functor on the category $\mathcal{C}$.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct.

If X is a set then we denote the power set of X by $P(X)$; the identity function on X is denoted by id_X.

If (X,τ) is a topological space and M is a subset of X, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of M in (X,τ) and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of M in (X,τ).

If $f : X \longrightarrow Y$ is a function and $M \subseteq X$ then $f_{\restriction M}$ is the restriction of f having M as a domain and $f(M)$ as a codomain. Further, we denote by $\mathbb{D}$ the set of all dyadic numbers of the interval $(0,1)$ and by $\mathbb{Q}$ the topological space of all rational numbers with their natural topology.

The closed maps and the open maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is *perfect* if it is closed and compact (i.e. point inverses are compact sets).

For all notions and notations not defined here see [10, 11, 13, 17].

1 Preliminaries

Definitions 1.1 An algebraic system $\underline{B} = (B, 0, 1, \vee, \wedge, ^*, C)$ is called a *contact Boolean algebra* or, briefly, *contact algebra* (abbreviated as CA) ([9]) if the system $(B, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra (where the operation "complement" is denoted by " $*$ ") and C is a binary relation on B, satisfying the following axioms:

(C1) If $a \neq 0$ then aCa;
(C2) If aCb then $a \neq 0$ and $b \neq 0$;
(C3) aCb implies bCa;
(C4) $aC(b \vee c)$ iff aCb or aCc.

We shall simply write (B, C) for a contact algebra. The relation C is called a *contact relation*. If $a \in B$ and $D \subseteq B$, we will write "aCD" for "$(\forall d \in D)(aCd)$". We will say that two CA's (B_1, C_1) and (B_2, C_2) are *CA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B_1 \longrightarrow B_2$ such that, for each $a, b \in B_1$, aC_1b iff $\varphi(a)C_2\varphi(b)$. A CA (B, C) is called a *complete contact Boolean algebra* or, briefly, *complete contact algebra* (abbreviated as CCA) if B is a complete Boolean algebra.

A contact algebra (B, C) is called a *normal contact Boolean algebra* or, briefly, *normal contact algebra* (abbreviated as NCA) ([3, 12]) if it satisfies the following axioms (we will write "$-C$" for "*not* C"):

(C5) If $a(-C)b$ then $a(-C)c$ and $b(-C)c^*$ for some $c \in B$;
(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

If an NCA is a CCA then it is called a *complete normal contact Boolean algebra* or, briefly, *complete normal contact algebra* (abbreviated as CNCA). The notion of normal contact algebra was introduced by Fedorchuk [12] under the name *Boolean δ-algebra* as an equivalent expression of the notion of *compingent Boolean algebra* of de Vries [3]. We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras and naturally arise in the normal Hausdorff spaces.

For any CA (B, C), we define a binary relation " $\ll_C$" on B (called *non-tangential inclusion*) by " $a \ll_C b \leftrightarrow a(-C)b^*$ ". Sometimes we will write simply " $\ll$" instead of " $\ll_C$".

Example 1.2 Let B be a Boolean algebra. Then there exist a largest and a smallest contact relations on B; the largest one, ρ_l, is defined by $a\rho_l b$ iff $a \neq 0$ and $b \neq 0$, and the smallest one, ρ_s, by $a\rho_s b$ iff $a \wedge b \neq 0$. Note that, for $a, b \in B$, $a \ll_{\rho_s} b$ iff $a \leq b$; hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus (B, ρ_s) is a normal contact algebra.

Example 1.3 Recall that a subset F of a topological space (X, τ) is called *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$. Clearly, F is regular closed iff it is the closure of an open set. For any topological space (X, τ), the collection $RC(X, \tau)$ (we will often write simply $RC(X)$) of all regular closed subsets of (X, τ) becomes a complete Boolean algebra $(RC(X, \tau), 0, 1, \wedge, \vee, {}^*)$ under the following operations: $1 = X, 0 = \emptyset, F^* = \mathrm{cl}(X \setminus F), F \vee G = F \cup G, F \wedge G = \mathrm{cl}(\mathrm{int}(F \cap G))$. The infinite operations are given by the following formulas: $\bigvee\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\bigcup\{F_\gamma \mid \gamma \in \Gamma\})$, and $\bigwedge\{F_\gamma \mid \gamma \in \Gamma\} = \mathrm{cl}(\mathrm{int}(\bigcap\{F_\gamma \mid \gamma \in \Gamma\}))$.

It is easy to see that setting $F\rho_{(X,\tau)}G$ iff $F \cap G \neq \emptyset$, we define a contact relation $\rho_{(X,\tau)}$ on $RC(X, \tau)$; it is called a *standard contact relation*. So, $(RC(X, \tau), \rho_{(X,\tau)})$ is a complete CA (it is called a *standard contact algebra*). We will often write simply ρ_X instead of $\rho_{(X,\tau)}$. Note that, for $F, G \in RC(X)$, $F \ll_{\rho_X} G$ iff $F \subseteq \mathrm{int}_X(G)$. Clearly, if (X, τ) is a normal Hausdorff space then the standard contact algebra $(RC(X, \tau), \rho_{(X,\tau)})$ is a complete NCA.

A subset U of a topological space (X, τ) is called *regular open* if $X \setminus U \in RC(X)$. The set of all regular open subsets of (X, τ) will be denoted by $RO(X, \tau)$ (or simply by $RO(X)$).

The next notion and assertion are inspired by the theory of proximity spaces (see, e.g., [17]):

1.4 Let (B, C) be a CA. Then a non-empty subset σ of B is called a *cluster in* (B, C) if the following conditions are satisfied:

(K1) If $a, b \in \sigma$ then aCb;
(K2) If $a \vee b \in \sigma$ then $a \in \sigma$ or $b \in \sigma$;
(K3) If aCb for every $b \in \sigma$, then $a \in \sigma$.

The set of all clusters in (B, C) will be denoted by $\mathrm{Clust}(B, C)$.

Theorem 1.5 ([22]) *A subset σ of a normal contact algebra (B, C) is a cluster iff there exists an ultrafilter u in B such that $\sigma = \{a \in B \mid aCb$ for every $b \in u\}$. Hence, if u is an ultrafilter in B then there exists a unique cluster σ_u in (B, C) containing u, and $\sigma_u = \{a \in B \mid aCb$ for every $b \in u\}$.*

The following notion is a lattice-theoretical counterpart of the Leader's notion of *local proximity* ([14]):

Definition 1.6 ([20]) An algebraic system $\underline{B}_l = (B, 0, 1, \vee, \wedge, ^*, \rho, \mathbb{B})$ is called a *local contact Boolean algebra* or, briefly, *local contact algebra* (abbreviated as LCA) if $(B, 0, 1, \vee, \wedge, ^*)$ is a Boolean algebra, ρ is a binary relation on B such that (B, ρ) is a CA, and $\mathbb{B}$ is an ideal (possibly non proper) of B, satisfying the following axioms:

(BC1) If $a \in \mathbb{B}$, $c \in B$ and $a \ll_\rho c$ then $a \ll_\rho b \ll_\rho c$ for some $b \in \mathbb{B}$;
(BC2) If $a\rho b$ then there exists an element c of $\mathbb{B}$ such that $a\rho(c \wedge b)$;
(BC3) If $a \neq 0$ then there exists $b \in \mathbb{B} \setminus \{0\}$ such that $b \ll_\rho a$.

We shall simply write $(B, \rho, \mathbb{B})$ for a local contact algebra. When B is a complete Boolean algebra, the LCA $(B, \rho, \mathbb{B})$ is called a *complete local contact Boolean algebra* or, briefly, *complete local contact algebra* (abbreviated as CLCA).

We will say that two local contact algebras $(B, \rho, \mathbb{B})$ and $(B_1, \rho_1, \mathbb{B}_1)$ are *LCA-isomorphic* if there exists a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for $a, b \in B$, $a\rho b$ iff $\varphi(a)\rho_1\varphi(b)$, and $\varphi(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$.

Note that if $(B, \rho, \mathbb{B})$ is a local contact algebra and $1 \in \mathbb{B}$ then (B, ρ) is a normal contact algebra. Conversely, any normal contact algebra (B, C) can be regarded as a local contact algebra of the form (B, C, B).

The following definitions and lemmas are lattice-theoretical counterparts of some notions and theorems from Leader's paper [14].

Definition 1.7 ([22]) Let $(B, \rho, \mathbb{B})$ be a local contact algebra. Define a binary relation "C_ρ" on B by $aC_\rho b$ iff $a\rho b$ or $a, b \notin \mathbb{B}$; it is called the *Alexandroff extension of* ρ.

Lemma 1.8 ([22]) *Let $(B, \rho, \mathbb{B})$ be a local contact algebra. Then (B, C_ρ), where C_ρ is the Alexandroff extension of ρ, is a normal contact algebra.*

Definition 1.9 Let $(B, \rho, \mathbb{B})$ be a local contact algebra. We will say that σ is a *cluster in* $(B, \rho, \mathbb{B})$ if σ is a cluster in the NCA (B, C_ρ). A cluster σ in $(B, \rho, \mathbb{B})$ (resp., an ultrafilter in B) is called *bounded* if $\sigma \cap \mathbb{B} \neq \emptyset$ (resp., $u \cap \mathbb{B} \neq \emptyset$). The set of all bounded clusters in $(B, \rho, \mathbb{B})$ will be denoted by $\mathrm{BClust}(B, \rho, \mathbb{B})$.

Lemma 1.10 ([22]) *Let $(B, \rho, \mathbb{B})$ be a local contact algebra and let $1 \notin \mathbb{B}$. Then $\sigma_\infty^{(B,\rho,\mathbb{B})} = \{b \in B \mid b \notin \mathbb{B}\}$ is a cluster in $(B, \rho, \mathbb{B})$. (Sometimes we will simply write σ_∞ instead of $\sigma_\infty^{(B,\rho,\mathbb{B})}$.)*

Notation 1.11 Let (X, τ) be a topological space. We denote by $CR(X, \tau)$ the family of all compact regular closed subsets of (X, τ). We will often write $CR(X)$ instead of $CR(X, \tau)$. If $x \in X$ then we set:

(1) $\qquad \sigma_x^X = \{F \in RC(X) \mid x \in F\}$ and $\nu_x^X = \{F \in RC(X) \mid x \in \mathrm{int}_X(F)\}$.

We will often write σ_x and ν_x instead of, respectively, σ_x^X and ν_x^X.

Fact 1.12 *([20, 22]) Let (X,τ) be a locally compact Hausdorff space. Then:*

(a) the triple $(RC(X,\tau),\rho_{(X,\tau)},CR(X,\tau))$ is a complete local contact algebra; it is called a standard local contact algebra*;*

(b) for every $x\in X$, σ_x is a bounded cluster in the standard local contact algebra $(RC(X,\tau),\rho_{(X,\tau)},CR(X,\tau))$ and ν_x is a filter in the Boolean algebra $RC(X)$.

The next theorem was proved by Roeper [20] (but its particular case concerning compact Hausdorff spaces and NCAs was proved by de Vries [3]). We will give a sketch of its proof; it follows the plan of the proof presented in [22]. The notations and the facts stated here will be used later on.

Theorem 1.13 (P. Roeper [20]) *There exists a bijective correspondence between the class of all (up to isomorphism) CLCAs and the class of all (up to homeomorphism) locally compact Hausdorff spaces; its restriction to the class of all (up to isomorphism) CNCAs gives a bijective correspondence between the later class and the class of all (up to homeomorphism) compact Hausdorff spaces.*

Sketch of the Proof. Let (X,τ) be a locally compact Hausdorff space. We put

$$\Psi^t(X,\tau)=(RC(X,\tau),\rho_{(X,\tau)},CR(X,\tau)).$$

Let $(B,\rho,\mathbb{B})$ be a complete local contact algebra. Let $C=C_\rho$ be the Alexandroff extension of ρ. Put $X=\mathrm{Clust}(B,C)$ and let $\mathfrak{T}$ be the topology on X having as a closed base the family $\{\lambda_{(B,C)}(a)\mid a\in B\}$ where, for every $a\in B$,

$$\lambda_{(B,C)}(a)=\{\sigma\in X\mid a\in\sigma\}.$$

Sometimes we will write simply λ_B instead of $\lambda_{(B,C)}$. Note that $X\setminus\lambda_B(a)=\mathrm{int}(\lambda_B(a^*))$, the family $\{\mathrm{int}(\lambda_B(a))\mid a\in B\}$ is an open base of $(X,\mathfrak{T})$ and, for every $a\in B$, $\lambda_B(a)\in RC(X,\mathfrak{T})$. Further,

$$\lambda_B:(B,C)\longrightarrow(RC(X),\rho_X)$$

is a CA-isomorphism and $(X,\mathfrak{T})$ is a compact Hausdorff space.

Let $1\in\mathbb{B}$. Then $C=\rho$ and $\mathbb{B}=B$, so that $(B,\rho,\mathbb{B})=(B,C,B)=(B,C)$ is a normal contact algebra and we put

$$\Psi^a(B,\rho,\mathbb{B})(=\Psi^a(B,C,B)=\Psi^a(B,C))=(X,\mathfrak{T}).$$

Let $1\notin\mathbb{B}$. Then we set $L=\mathrm{BClust}(B,\rho,\mathbb{B})(=X\setminus\{\sigma_\infty\})$ (sometimes we will write $L_{(B,\rho,\mathbb{B})}$ or L_B instead of L). Let the topology $\tau(=\tau_{(B,\rho,\mathbb{B})})$ on L be the subspace topology, i.e. $\tau=\mathfrak{T}_{|_L}$. Then (L,τ) is a locally compact Hausdorff space. We put

$$\Psi^a(B,\rho,\mathbb{B})=(L,\tau).$$

Let $\lambda^l_{(B,\rho,\mathbb{B})}(a)=\lambda_{(B,C_\rho)}(a)\cap L$, for each $a\in B$. We will write simply λ^l_B (or even $\lambda_{(B,\rho,\mathbb{B})}$) instead of $\lambda^l_{(B,\rho,\mathbb{B})}$ when this does not lead to ambiguity. Then L is a dense

subset of the topological space X and $\lambda^l_B : (B, \rho, \mathbb{B}) \longrightarrow (RC(L), \rho_L, CR(L))$ is an LCA-isomorphism. Note also that for every $b \in B$, $\mathrm{int}_{L_B}(\lambda^l_B(b)) = L_B \cap \mathrm{int}_X(\lambda_B(b))$ and $L \setminus \lambda^l_B(b) = \mathrm{int}_L(\lambda^l_B(b^*))$.

For every CLCA $(B, \rho, \mathbb{B})$ and every $a \in B$, set

$$\lambda^g_{(B,\rho,\mathbb{B})}(a) = \lambda_{(B,C_\rho)}(a) \cap \Psi^a(B, \rho, \mathbb{B}).$$

We will write simply λ^g_B instead of $\lambda^g_{(B,\rho,\mathbb{B})}$ when this does not lead to ambiguity. Thus, when $1 \in \mathbb{B}$, we have that $\lambda^g_B = \lambda_B$, and if $1 \notin \mathbb{B}$ then $\lambda^g_B = \lambda^l_B$. Hence,

$$\lambda^g_B : (B, \rho, \mathbb{B}) \longrightarrow (\Psi^t \circ \Psi^a)(B, \rho, \mathbb{B})$$

is and an LCA-isomorphism. Let (L, τ) be a locally compact Hausdorff space. Then the map

$$t_{(L,\tau)} : (L, \tau) \longrightarrow \Psi^a(\Psi^t(L, \tau)),$$

defined by $t_{(L,\tau)}(x) = \sigma_x$, for all $x \in L$, is a homeomorphism; we will often write simply t_L instead of $t_{(L,\tau)}$. Therefore $\Psi^a(\Psi^t(L, \tau))$ is homeomorphic to (L, τ) and $\Psi^t(\Psi^a(B, \rho, \mathbb{B}))$ is LCA-isomorphic to $(B, \rho, \mathbb{B})$. $\square$

Note that if $(B, \rho, \mathbb{B})$ is an LCA, then for every $a \in B$, $a = \bigvee\{b \in \mathbb{B} \mid b \ll_\rho a\}$.

We will need also the following assertion from [5]:

Proposition 1.14 ([5]) *Let $(A, \rho, \mathbb{B})$ be an LCA and σ_1, σ_2 be two clusters in $(A, \rho, \mathbb{B})$ such that $\mathbb{B} \cap \sigma_1 = \mathbb{B} \cap \sigma_2$. Then $\sigma_1 = \sigma_2$.*

We will recall some results from [5, 4] which are basic for our investigations in the present paper.

Definitions 1.15 ([5]) Let **HLC** be the category of all locally compact Hausdorff spaces and all continuous maps between them.

Let **DHLC** be the category whose objects are all complete LCAs and whose morphisms are all functions $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ between the objects of **DHLC** satisfying conditions

(DLC1) $\varphi(0) = 0$;
(DLC2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;
(DLC3) If $a \in \mathbb{B}, b \in A$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$;
(DLC4) For every $b \in \mathbb{B}'$ there exists $a \in \mathbb{B}$ such that $b \leq \varphi(a)$;
(DLC5) $\varphi(a) = \bigvee\{\varphi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$, for every $a \in A$;

let the composition "$\diamond$" of two morphisms $\varphi_1 : (A_1, \rho_1, \mathbb{B}_1) \longrightarrow (A_2, \rho_2, \mathbb{B}_2)$ and $\varphi_2 : (A_2, \rho_2, \mathbb{B}_2) \longrightarrow (A_3, \rho_3, \mathbb{B}_3)$ of **DHLC** be defined by the formula $\varphi_2 \diamond \varphi_1 = (\varphi_2 \circ \varphi_1)\check{}$, where, for every function $\psi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ between two objects of **DHLC**, $\psi\check{} : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ is defined as follows: $\psi\check{}(a) = \bigvee\{\psi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$, for every $a \in A$.

As it was shown in [5], condition (DLC3) can be replaced by the following one:
(DLC3S) If $a, b \in A$ and $a \ll_\rho b$, then $(\varphi(a^*))^* \ll_\eta \varphi(b)$.

We will need the following duality theorem:

Theorem 1.16 ([5]) *The categories* **HLC** *and* **DHLC** *are dually equivalent. In more details, let* $\Lambda^t : \mathbf{HLC} \longrightarrow \mathbf{DHLC}$ *and* $\Lambda^a : \mathbf{DHLC} \longrightarrow \mathbf{HLC}$ *be the contravariant functors extending, respectively, the Roeper's correspondences* Ψ^t *and* Ψ^a *(see Theorem 1.13) to the morphisms of the categories* **HLC** *and* **DHLC** *in the following way: for every* $f \in \mathbf{HLC}(X, Y)$ *and every* $G \in RC(Y)$*,*

$$\Lambda^t(f)(G) = \mathrm{cl}(f^{-1}(\mathrm{int}(G))),$$

and for every $\varphi \in \mathbf{DHLC}((A, \rho, \mathbb{B}), (B, \eta, \mathbb{B}'))$ *and for every* $\sigma' \in \Lambda^a(B, \eta, \mathbb{B}')$*,*

$$\Lambda^a(\varphi)(\sigma') \cap \mathbb{B} = \{a \in \mathbb{B} \mid \textit{if } b \in A \textit{ and } a \ll_\rho b \textit{ then } \varphi(b) \in \sigma'\} \tag{2}$$

(if, in addition, φ *is a complete Boolean homomorphism, then the above formula is equivalent to the following one: for every bounded ultrafilter* u *in* B*,* $\Lambda^a(\varphi)(\sigma_u) = \sigma_{\varphi^{-1}(u)}$*); then* $\lambda^g : Id_{\mathbf{DHLC}} \longrightarrow \Lambda^t \circ \Lambda^a$*, where* $\lambda^g(A, \rho, \mathbb{B}) = \lambda^g_A$ *for every* $(A, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$ *, and* $t^l : Id_{\mathbf{HLC}} \longrightarrow \Lambda^a \circ \Lambda^t$*, where* $t^l(X) = t_X$ *for every* $X \in |\mathbf{HLC}|$*, are natural isomorphisms.*

Definition 1.17 ([4]) Let **OHLC** be the category of all locally compact Hausdorff spaces and all open maps between them.

Let **DOHLC** be the subcategory of the category **DHLC** having the same objects and whose morphisms are all **DHLC**-morphisms $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ which are complete Boolean homomorphisms and satisfy the following condition:
(LO) $\forall a \in A$ and $\forall b \in \mathbb{B}'$, $\varphi_\Lambda(b)\rho a$ implies $b\eta\varphi(a)$,
where φ_Λ is the left adjoint of φ (i.e. $\varphi_\Lambda : B \longrightarrow A$ is an order-preserving map such that $\forall b \in B$, $\varphi(\varphi_\Lambda(b)) \geq b$ and $\forall a \in A$, $\varphi_\Lambda(\varphi(a)) \leq a$; its existence follows from the Adjoint Functor Theorem (see, e.g., [13])).

Theorem 1.18 ([4]) *The categories* **OHLC** *and* **DOHLC** *are dually equivalent.*

Finally, we will recall some definitions and facts from the theory of extensions of topological spaces, as well as the fundamental Leader Local Compactification Theorem [14].

Let X be a Tychonoff space. We will denote by $\mathcal{L}(X)$ the set of all, up to equivalence, locally compact Hausdorff extensions of X (recall that two (locally compact Hausdorff) extensions (Y_1, f_1) and (Y_2, f_2) of X are said to be *equivalent* iff there exists a homeomorphism $h : Y_1 \longrightarrow Y_2$ such that $h \circ f_1 = f_2$); the equivalence class of (Y, f) will be denoted by $[(Y, f)]$. Let $[(Y_i, f_i)] \in \mathcal{L}(X)$, where $i = 1, 2$. We set $[(Y_1, f_1)] \leq [(Y_2, f_2)]$ if there exists a continuous mapping $h : Y_2 \longrightarrow Y_1$ such that $f_1 = h \circ f_2$. Then $(\mathcal{L}(X), \leq)$ is a poset.

Let X be a Tychonoff space. We will denote by $\mathcal{K}(X)$ the set of all, up to equivalence, Hausdorff compactifications of X.

Recall that if X is a set and $P(X)$ is the power set of X ordered by the inclusion (and thus $P(X)$ becomes a Boolean algebra), then a triple $(X, \beta, \mathcal{B})$ is called a *local proximity space* (see [14]) if $(P(X), \beta)$ is a CA, $\mathcal{B}$ is an ideal (possibly non proper) of $P(X)$ and the axioms (BC1),(BC2) from 1.6 are fulfilled. A local proximity space $(X, \beta, \mathcal{B})$ is said to be *separated* if β is the identity relation on singletons. Recall that every separated local proximity space $(X, \beta, \mathcal{B})$ induces a Tychonoff topology $\tau_{(X,\beta,\mathcal{B})}$ in X by defining $\mathrm{cl}(M) = \{x \in X \mid x\beta M\}$ for every $M \subseteq X$ ([14]). If (X, τ) is a topological space then we say that $(X, \beta, \mathcal{B})$ is a *local proximity space on* (X, τ) if $\tau_{(X,\beta,\mathcal{B})} = \tau$.

The set of all separated local proximity spaces on a Tychonoff space (X, τ) will be denoted by $\mathcal{LP}(X, \tau)$. A partial order in $\mathcal{LP}(X, \tau)$ is defined by $(X, \beta_1, \mathcal{B}_1) \preceq (X, \beta_2, \mathcal{B}_2)$ if $\beta_2 \subseteq \beta_1$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$ (see [14]).

A function $f : X_1 \longrightarrow X_2$ between two local proximity spaces $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$ is said to be an *equicontinuous mapping* (see [14]) if the following two conditions are fulfilled:

(EQ1) $A\beta_1 B$ implies $f(A)\beta_2 f(B)$, for $A, B \subseteq X$, and

(EQ2) $B \in \mathcal{B}_1$ implies $f(B) \in \mathcal{B}_2$.

The separated local proximity spaces of the form $(X, \delta, P(X))$ are denoted by (X, δ) and are called *Efremovič proximity spaces*. The equicontinuous mappings between Efremovič proximity spaces are called *proximally continuous mappings*.

Theorem 1.19 (S. Leader [14]) *Let (X, τ) be a Tychonoff space. Then there exists an isomorphism Λ_X between the ordered sets $(\mathcal{L}(X, \tau), \leq)$ and $(\mathcal{LP}(X, \tau), \preceq)$. In more details, for every $(X, \beta, \mathcal{B}) \in \mathcal{LP}(X, \tau)$ there exists a locally compact Hausdorff extension (Y, f) of X satisfying the following two conditions:*

(a) $A\beta B$ iff $\mathrm{cl}_Y(f(A)) \cap \mathrm{cl}_Y(f(B)) \neq \emptyset$;

(b) $B \in \mathcal{B}$ iff $\mathrm{cl}_Y(f(B))$ is compact.

Such a local compactification is unique up to equivalence; we set $(Y, f) = L(X, \beta, \mathcal{B})$ and $(\Lambda_X)^{-1}(X, \beta, \mathcal{B}) = [(Y, f)]$. The space Y is compact iff $X \in \mathcal{B}$. Conversely, if (Y, f) is a locally compact Hausdorff extension of X and β and $\mathcal{B}$ are defined by (a) and (b), then $(X, \beta, \mathcal{B})$ is a separated local proximity space, and we set $\Lambda_X([(Y, f)]) = (X, \beta, \mathcal{B})$.

Let $(X_i, \beta_i, \mathcal{B}_i)$, $i = 1, 2$, be two separated local proximity spaces and $f : X_1 \longrightarrow X_2$ be a function. Let $(Y_i, f_i) = L(X_i, \beta_i, \mathcal{B}_i)$, where $i = 1, 2$. Then there exists a continuous map $L(f) : Y_1 \longrightarrow Y_2$ such that $f_2 \circ f = L(f) \circ f_1$ iff f is an equicontinuous map between $(X_1, \beta_1, \mathcal{B}_1)$ and $(X_2, \beta_2, \mathcal{B}_2)$.

We will also need a lemma from [2]:

Lemma 1.20 *Let X be a dense subspace of a topological space Y. Then the functions $r : RC(Y) \longrightarrow RC(X)$, $F \mapsto F \cap X$, and $e : RC(X) \longrightarrow RC(Y)$, $G \mapsto$*

$\mathrm{cl}_Y(G)$*, are Boolean isomorphisms between Boolean algebras* $RC(X)$ *and* $RC(Y)$*, and* $e \circ r = id_{RC(Y)}$*,* $r \circ e = id_{RC(X)}$*.*

2 A de Vries-type revision of the Leader Local Compactification Theorem

In this section we will obtain a strengthening of Leader Local Compactification Theorem ([14]); it is similar to de Vries' ([3]) strengthening of Smirnov Compactification Theorem ([21]).

Definition 2.1 Let (X,τ) be a Tychonoff space. An LCA $(RC(X,\tau),\rho,\mathbb{B})$ is said to be *admissible for* (X,τ) if it satisfies the following conditions:

(A1) if $F, G \in RC(X)$ and $F \cap G \neq \emptyset$ then $F\rho G$;

(A2) if $F \in RC(X)$ and $x \in \mathrm{int}_X(F)$ then there exists $G \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G)$ and $G \ll_\rho F$.

The set of all LCAs $(RC(X,\tau),\rho,\mathbb{B})$ which are admissible for (X,τ) will be denoted by $\mathfrak{L}_{ad}(X,\tau)$ (or simply by $\mathfrak{L}_{ad}(X)$). If $(RC(X),\rho_i,\mathbb{B}_i) \in \mathfrak{L}_{ad}(X)$, where $i = 1,2$, then we set $(RC(X),\rho_1,\mathbb{B}_1) \preceq_{ad} (RC(X),\rho_2,\mathbb{B}_2)$ iff $\rho_2 \subseteq \rho_1$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Obviously, $(\mathfrak{L}_{ad}(X,\tau),\preceq_{ad})$ is a poset.

Theorem 2.2 *Let* (X,τ) *be a Tychonoff space. Then the posets* $(\mathfrak{L}(X,\tau),\leq)$ *and* $(\mathfrak{L}_{ad}(X,\tau),\preceq_{ad})$ *are isomorphic.*

Proof. Let (Y,f) be a locally compact Hausdorff extensions of X. Set

(3) $\quad \mathbb{B}_{(Y,f)} = f^{-1}(CR(Y))$ and let $F\eta_{(Y,f)}G \iff \mathrm{cl}_Y(f(F)) \cap \mathrm{cl}_Y(f(G)) \neq \emptyset$,

for every $F, G \in RC(X)$. Note that, by 1.20, $\mathbb{B}_{(Y,f)} = \{F \in RC(X) \mid \mathrm{cl}_Y(f(F))$ is compact$\}$. Hence $\mathbb{B}_{(Y,f)} \subseteq RC(X)$. We will show that $(RC(X),\eta_{(Y,f)},\mathbb{B}_{(Y,f)}) \in \mathfrak{L}_{ad}(X)$. We have, by 1.20, that the map

(4) $\quad r_{(Y,f)} : (RC(Y),\rho_Y,CR(Y)) \longrightarrow (RC(X),\eta_{(Y,f)},\mathbb{B}_{(Y,f)}),\ G \mapsto f^{-1}(G),$

is a Boolean isomorphism and, for every $F, G \in RC(Y)$, the following is fulfilled: $F\rho_{(Y,f)}G$ iff $r_{(Y,f)}(F)\eta_{(Y,f)}r_{(Y,f)}(G)$, and $F \in CR(Y)$ iff $r_{(Y,f)}(F) \in \mathbb{B}_{(Y,f)}$. Hence $(RC(X),\eta_{(Y,f)},\mathbb{B}_{(Y,f)})$ is an LCA and $r_{(Y,f)}$ is an LCA-isomorphism. Clearly, condition (A1) is fulfilled. Let now $F \in RC(X)$. Set $U = \mathrm{int}_X(F)$ and let $x \in U$. There exists an open subset V of Y such that $V \cap f(X) = f(U)$. Since Y is a locally compact Hausdorff space, there exists an $H \in CR(Y)$ with $f(x) \in \mathrm{int}_Y(H) \subseteq H \subseteq V$. Let $G = f^{-1}(H)$. Then $H \in \mathbb{B}_{(Y,f)}$ and, obviously, $x \in \mathrm{int}_X(G)$ and $G \ll_{\eta_Y} F$. So, condition (A2) is also checked. Hence $(RC(X),\eta_{(Y,f)},\mathbb{B}_{(Y,f)}) \in \mathfrak{L}_{ad}(X)$. It is clear

that if (Y_1, f_1) is a locally compact Hausdorff extensions of X equivalent to the extension (Y, f), then $(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) = (RC(X), \eta_{(Y_1,f_1)}, \mathbb{B}_{(Y_1,f_1)})$. Therefore, a map

(5) $$\alpha_X : \mathcal{L}(X) \longrightarrow \mathcal{L}_{ad}(X),\ [(Y, f)] \mapsto (RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}),$$

is well-defined.

Set, for short, $A = RC(X)$. Let $(A, \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X)$ and $Y = \Lambda^a(A, \rho, \mathbb{B})$. Then, by Roeper's Theorem 1.13, Y is a locally compact Hausdorff space. Let us show that for every $x \in X$, we have that $\sigma_x \in Y$ (where $\sigma_x = \{F \in A \mid x \in F\}$). By 1.12, ν_x is a filter in the Boolean algebra A. Hence there exists an ultrafilter u in A such that $\nu_x \subseteq u$. It is easy to see that $u \subseteq \sigma_x$. Let $\sigma = \{F \in A \mid FC_\rho u\}$ (i.e. $\sigma = \sigma_u$). Since, by (A2), $\nu_x \cap \mathbb{B} \neq \emptyset$, we get that $\sigma \in Y$. We will show that $\sigma_x = \sigma$. Indeed, let $F \in \sigma_x$ and $G \in u$. Then $x \in F \cap G$. Thus, by (A1), $F\rho G$. This implies that $FC_\rho u$, i.e. that $F \in \sigma$. So, $\sigma_x \subseteq \sigma$. Now, suppose that there exists $F \in \sigma$ such that $x \notin F$. Then $x \in X \setminus F = \mathrm{int}_X(F^*)$. Thus, by (A2), there exists $G \in \mathbb{B}$ such that $x \in \mathrm{int}_X(G)$ and $G \ll_\rho F^*$. Therefore $G \in \nu_x$ and $G(-\rho)F$. Since $G \in \mathbb{B}$, we get that $F(-C_\rho)G$, a contradiction. So, we have proved that $\sigma_x = \sigma$ and, thus, $\sigma_x \in Y$ for every $x \in X$. Define

(6) $$f_{(\rho,\mathbb{B})} : X \longrightarrow Y,\ x \mapsto \sigma_x.$$

Set, for short, $f = f_{(\rho,\mathbb{B})}$. Then $\mathrm{cl}_Y(f(X)) = Y$. Indeed, for every $F \in \mathbb{B} \setminus \{\emptyset\}$ and for every $x \in F$, we have that $\sigma_x \in f(X) \cap \lambda^g_A(F)$. Since Y is regular, this implies that $\mathrm{cl}_Y(f(X)) = Y$. We will now show that f is a homeomorphic embedding. It is clear that f is an injection. Further, let $x \in X$, $F \in \mathbb{B}$ and $\sigma_x \in \mathrm{int}_Y(\lambda^g_A(F))$. Since $\mathrm{int}_Y(\lambda^g_A(F)) = Y \setminus \lambda^g_A(F^*)$, we get that $\sigma_x \notin \lambda^g_A(F^*)$. Thus $F^* \notin \sigma_x$. This implies that $x \notin F^*$, i.e. $x \in X \setminus F^* = \mathrm{int}_X(F)$. Moreover, $f(\mathrm{int}_X(F)) \subseteq \mathrm{int}_Y(\lambda^g_A(F))$. Indeed, if $y \in \mathrm{int}_X(F)$ then $y \notin F^*$; thus $F^* \notin \sigma_y$, i.e. $\sigma_y \notin \lambda^g_A(F^*)$; this implies that $\sigma_y \in \mathrm{int}_Y(\lambda^g_A(F))$. All this shows that f is a continuous function. Set $g = ((f)_{\restriction X})^{-1}$, where $(f)_{\restriction X} : X \longrightarrow f(X)$ is the restriction of f. We have to show that g is a continuous function. Let $x \in X$, $F \in A$ and $x \in \mathrm{int}_X(F)$. We have that $x = g(\sigma_x)$. Let $\sigma_y \in \mathrm{int}_Y(\lambda^g_A(F))$. Then $\sigma_y \in Y \setminus \lambda^g_A(F^*)$, i.e. $y \notin F^*$; thus $y \in X \setminus F^* = \mathrm{int}_X(F)$. Therefore, $g(\mathrm{int}_Y(\lambda^g_A(F))) \subseteq \mathrm{int}_X(F)$. So, g is a continuous function. All this shows that (Y, f) is a locally compact Hausdorff extension of X. We now set:

(7) $$\beta_X : \mathcal{L}_{ad}(X) \longrightarrow \mathcal{L}(X),\ (RC(X), \rho, \mathbb{B}) \mapsto [(\Lambda^a(RC(X), \rho, \mathbb{B}), f_{(\rho,\mathbb{B})})].$$

We will show that $\alpha_X \circ \beta_X = id_{\mathcal{L}_{ad}(X)}$ and $\beta_X \circ \alpha_X = id_{\mathcal{L}(X)}$.

Let $[(Y, f)] \in \mathcal{L}(X)$. Then $\beta_X(\alpha_X([(Y, f)])) = \beta_X(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}) = [(\Lambda^a(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)}), f_{(\eta_{(Y,f)},\mathbb{B}_{(Y,f)})})]$. Set, for short, $\eta = \eta_{(Y,f)}$, $\mathbb{B} = \mathbb{B}_{(Y,f)}$, $g = f_{(\eta_{(Y,f)},\mathbb{B}_{(Y,f)})}$, $Z = \Lambda^a(RC(X), \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$ and $r_{(Y,f)} = r_f$. We have to show that $[(Y, f)] = [(Z, g)]$. Since r_f is an LCA-isomorphism, we get that $h = \Lambda^a(r_f) : Z \longrightarrow \Lambda^a(\Lambda^t(Y))$ is a homeomorphism. Set $Y' = \Lambda^a(\Lambda^t(Y))$. By Roeper's Theorem 1.13, the map $t_Y : Y \longrightarrow Y'$, $y \mapsto \sigma_y$ is a homeomorphism. Let $h' = (t_Y)^{-1} \circ h$.

Then $h' : Z \longrightarrow Y$ is a homeomorphism. We will prove that $h' \circ g = f$ and this will imply that $[(Y, f)] = [(Z, g)]$. Let $x \in X$ and u be an ultrafilter containing the filter ν_x. Then, as we have shown above, $\sigma_x = \sigma_u$. Hence $h(\sigma_x) = h(\sigma_u) = \sigma_{e_f(u)}$, where $e_f = (r_f)^{-1}$. Thus $h'(g(x)) = h'(\sigma_x) = (t_Y)^{-1}(h(\sigma_x)) = (t_Y)^{-1}(\sigma_{e_f(u)})$. Note that, by 1.20, $e_f(F) = \mathrm{cl}_Y(f(F))$, for every $F \in RC(X)$. Since $e_f : RC(X) \longrightarrow RC(Y)$ is a Boolean isomorphism, we get that $e_f(u)$ is an ultrafilter in $RC(Y)$ containing $\nu^Y_{f(x)}$. Thus $\sigma_{e_f(u)} = \sigma^Y_{f(x)}$. Hence $(t_Y)^{-1}(\sigma_{e_f(u)}) = f(x)$. So, $h' \circ g = f$. Therefore, $\beta_X \circ \alpha_X = id_{\mathfrak{L}(X)}$.

Let $(RC(X), \rho, \mathbb{B}) \in \mathfrak{L}_{ad}(X)$ and $Y = \Lambda^a(RC(X), \rho, \mathbb{B})$. Recall that we have set $A = RC(X)$. We have that $\beta_X(A, \rho, \mathbb{B}) = [(Y, f_{(\rho,\mathbb{B})})]$. Set $f = f_{(\rho,\mathbb{B})}$. Then $\alpha_X(\beta_X(A, \rho, \mathbb{B})) = (A, \eta_{(Y,f)}, \mathbb{B}_{(Y,f)})$. By Roeper's Theorem 1.13, we have that $\lambda^g_A : (A, \rho, \mathbb{B}) \longrightarrow (RC(Y), \rho_Y, CR(Y))$ is an LCA-isomorphism. We will show that $f^{-1}(\lambda^g_A(F)) = F$, for every $F \in RC(X)$. Indeed, if $x \in F$ then $F \in \sigma_x$, and thus $\sigma_x \in \lambda^g_A(F)$; hence $f(F) \subseteq \lambda^g_A(F)$, i.e. $F \subseteq f^{-1}(\lambda^g_A(F))$. If $x \in f^{-1}(\lambda^g_A(F))$ then $f(x) \in \lambda^g_A(F)$, i.e. $\sigma_x \in \lambda^g_A(F)$; therefore $F \in \sigma_x$, which means that $x \in F$. So, $f^{-1}(\lambda^g_A(F)) = F$, for every $F \in RC(X)$. Since $CR(Y) = \{\lambda^g_A(F) \mid F \in \mathbb{B}\}$, we get that $f^{-1}(CR(Y)) = \mathbb{B}$. Thus $\mathbb{B}_{(Y,f)} = \mathbb{B}$. Further, by 1.20, $\mathrm{cl}_Y(f(F)) = \lambda^g_A(F)$, for every $F \in RC(X)$. Since, for every $F, G \in RC(X)$, $F \rho G \iff \lambda^g_A(F) \cap \lambda^g_A(G) \neq \emptyset$, we get that $\rho = \eta_{(Y,f)}$. Therefore, $\alpha_X \circ \beta_X = id_{\mathfrak{L}_{ad}(X)}$.

We will now prove that α_X and β_X are monotone maps.

Let $[(Y_i, f_i)] \in \mathfrak{L}(X)$, where $1 = 1, 2$, and $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. Then there exists a continuous map $g : Y_2 \longrightarrow Y_1$ such that $g \circ f_2 = f_1$. Let $\alpha_X([(Y_i, f_i)]) = (RC(X), \eta_{(Y_i,f_i)}, \mathbb{B}_{(Y_i,f_i)})$, where $i = 1, 2$. Set $\eta_i = \eta_{(Y_i,f_i)}$ and $\mathbb{B}_i = \mathbb{B}_{(Y_i,f_i)}$, $i = 1, 2$. We have to show that $\eta_2 \subseteq \eta_1$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F \in \mathbb{B}_2$. Then $\mathrm{cl}_{Y_2}(f_2(F))$ is compact. Hence $g(\mathrm{cl}_{Y_2}(f_2(F)))$ is compact. We have that $f_1(F) = g(f_2(F)) \subseteq g(\mathrm{cl}_{Y_2}(f_2(F))) \subseteq \mathrm{cl}_{Y_1}(g(f_2(F))) = \mathrm{cl}_{Y_1}(f_1(F))$. Thus $\mathrm{cl}_{Y_1}(f_1(F)) = g(\mathrm{cl}_{Y_2}(f_2(F)))$, i.e. $\mathrm{cl}_{Y_1}(f_1(F))$ is compact. Therefore $F \in \mathbb{B}_1$. So, we have proved that $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F, G \in RC(X)$ and $F \eta_2 G$. Then there exists $y \in \mathrm{cl}_{Y_2}(f_2(F)) \cap \mathrm{cl}_{Y_2}(f_2(G))$. Since $g(\mathrm{cl}_{Y_2}(f_2(F))) \subseteq \mathrm{cl}_{Y_1}(f_1(F))$ and, analogously, $g(\mathrm{cl}_{Y_2}(f_2(G))) \subseteq \mathrm{cl}_{Y_1}(f_1(G))$, we get that $g(y) \in \mathrm{cl}_{Y_1}(f_1(F)) \cap \mathrm{cl}_{Y_1}(f_1(G))$. Thus $F \eta_1 G$. Therefore, $\eta_2 \subseteq \eta_1$. All this shows that $\alpha_X([(Y_1, f_1)]) \preceq_{ad} \alpha_X([(Y_2, f_2)])$. Hence, α_X is a monotone function.

Let now $(RC(X), \rho_i, \mathbb{B}_i) \in \mathfrak{L}_{ad}(X)$, where $i = 1, 2$, and $(RC(X), \rho_1, \mathbb{B}_1) \preceq_{ad} (RC(X), \rho_2, \mathbb{B}_2)$. Set, for short, $Y_i = \Lambda^a(RC(X), \rho_i, \mathbb{B}_i)$ and $f_i = f_{(\rho_i,\mathbb{B}_i)}$, $i = 1, 2$. Then $\beta_X(RC(X), \rho_i, \mathbb{B}_i) = [(Y_i, f_i)]$, $i = 1, 2$. We will show that $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. We have that $f_i : X \longrightarrow Y_i$ is defined by $f_i(x) = \sigma_x$, for every $x \in X$ and $i = 1, 2$. We also have that $\mathbb{B}_2 \subseteq \mathbb{B}_1$ and $\rho_2 \subseteq \rho_1$. Let us regard the following function $\varphi : (RC(X), \rho_1, \mathbb{B}_1) \longrightarrow (RC(X), \rho_2, \mathbb{B}_2)$, $F \mapsto F$. We will prove that φ is a **DHLC**-morphism. Clearly, φ satisfies conditions (DLC1) and (DLC2). The fact that $\rho_2 \subseteq \rho_1$ implies immediately that φ satisfies also condition (DLC3). Further, for establishing condition (DLC4) use the fact that $\mathbb{B}_2 \subseteq \mathbb{B}_1$. Let $F \in RC(X)$. Then $F = \bigvee\{G \in \mathbb{B}_1 \mid G \ll_{\rho_1} F\}$ and thus $\varphi(F) = \bigvee\{\varphi(G) \mid G \in \mathbb{B}_1, G \ll_{\rho_1} F\}$. This shows that φ satisfies condition (DLC5). So, φ is a **DHLC**-morphism. Then $g = \Lambda^a(\varphi) : Y_2 \longrightarrow Y_1$ is a continuous map. We will prove that $g \circ f_2 = f_1$,

i.e. that for every $x \in X$, $g(\sigma_x) = \sigma_x$. So, let $x \in X$. We have, by (2), that $g(\sigma_x) \cap \mathbb{B}_1 = \{F \in \mathbb{B}_1 \mid (\forall G \in RC(X))[(F \ll_{\rho_1} G) \to (x \in G)]\}$. We will show that $g(\sigma_x) \cap \mathbb{B}_1 = \sigma_x \cap \mathbb{B}_1$. This will imply, by 1.14, that $g(\sigma_x) = \sigma_x$. Let $F \in \sigma_x \cap \mathbb{B}_1$. Then $x \in F$ and thus $F \in g(\sigma_x) \cap \mathbb{B}_1$. Conversely, suppose that there exists $H \in g(\sigma_x) \cap \mathbb{B}_1$ such that $x \notin H$. Then $x \in X \setminus H = \mathrm{int}_X(H^*)$. By (A2), there exists $G \in \mathbb{B}_1$ with $x \in \mathrm{int}_X(G)$ and $G \ll_{\rho_1} H^*$. We get that $H \ll_{\rho_1} G^*$ and $x \notin G^*$, a contradiction. Therefore, $g(\sigma_x) = \sigma_x$. Thus $[(Y_1, f_1)] \leq [(Y_2, f_2)]$. So, β_X is also a monotone function. Since $\beta_X = (\alpha_X)^{-1}$, we get that α_X is an isomorphism. □

Definition 2.3 Let (X, τ) be a Tychonoff space. An NCA $(RC(X, \tau), C)$ is said to be *admissible for* (X, τ) if the LCA $(RC(X, \tau), C, RC(X, \tau)) \in \mathcal{L}_{ad}(X, \tau)$. The set of all NCAs which are admissible for (X, τ) will be denoted by $\mathcal{K}_{ad}(X, \tau)$ (or simply by $\mathcal{K}_{ad}(X)$). Note that $\mathcal{K}_{ad}(X)$ is, in fact, a subset of $\mathcal{L}_{ad}(X)$. The restriction on $\mathcal{K}_{ad}(X)$ of the order $\preceq_{ad}$, defined on $\mathcal{L}_{ad}(X)$, will be denoted again by $\preceq_{ad}$.

Corollary 2.4 (de Vries [3]) *For every Tychonoff space X, there exists an isomorphism between the posets $(\mathcal{K}(X), \leq)$ and $(\mathcal{K}_{ad}(X), \preceq_{ad})$.*

Proof. It follows immediately from Theorem 2.2. □

The first part of Leader Local Compactification Theorem 1.19 follows from our Theorem 2.2 and the following three lemmas.

Lemma 2.5 *Let $(X, \beta_i, \mathcal{B}_i)$, $i = 1, 2$, be two separated local proximity spaces on a Tychonoff space (X, τ), $\mathcal{B}_1 \cap RC(X) = \mathcal{B}_2 \cap RC(X)$ and $(\beta_1)_{|RC(X)} = (\beta_2)_{|RC(X)}$ (i.e., for every $F, G \in RC(X)$, $F\beta_1 G \iff F\beta_2 G$). Then $\beta_1 = \beta_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.*

Proof. Let, for $i = 1, 2$, $(Y_i, f_i) = L(X, \beta_i, \mathcal{B}_i)$ (see Theorem 1.19). Let $B \in \mathcal{B}_1$. Then $\mathrm{cl}_{Y_1}(f_1(B))$ is compact. There exists an open subset U of Y_1 such that $\mathrm{cl}_{Y_1}(f_1(B)) \subseteq U$ and $\mathrm{cl}_{Y_1}(U)$ is compact. Let $F = f_1^{-1}(\mathrm{cl}_{Y_1}(U))$. Then $F \in RC(X)$ and $\mathrm{cl}_{Y_1}(f_1(F)) = \mathrm{cl}_{Y_1}(U)$. Hence $F \in \mathcal{B}_1 \cap RC(X)$. Thus $F \in \mathcal{B}_2$. Since $B \subseteq F$, we get that $B \in \mathcal{B}_2$. Therefore, $\mathcal{B}_1 \subseteq \mathcal{B}_2$. Analogously we obtain that $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Thus $\mathcal{B}_1 = \mathcal{B}_2$.

Let $M, N \subseteq X$ and $M(-\beta_1)N$. Suppose that $M\beta_2 N$. Then there exist $M', N' \in \mathcal{B}_2$ such that $M' \subseteq M$, $N' \subseteq N$ and $M'\beta_2 N'$. Since $\mathcal{B}_1 = \mathcal{B}_2$, we get that $M', N' \in \mathcal{B}_1$. Hence $K_1 = \mathrm{cl}_{Y_1}(f_1(M'))$ and $K_2 = \mathrm{cl}_{Y_1}(f_1(N'))$ are disjoint compact subsets of Y_1. Then there exist open subsets U and V of Y_1 having disjoint closures in Y_1 and containing, respectively, K_1 and K_2. Set $F = f_1^{-1}(\mathrm{cl}_{Y_1}(U))$ and $G = f_1^{-1}(\mathrm{cl}_{Y_1}(V))$. Then $F, G \in RC(X)$, $M' \subseteq F$, $N' \subseteq G$ and $F(-\beta_1)G$. Thus $F(-\beta_2)G$ and hence $M'(-\beta_2)N'$, a contradiction. Therefore, $M(-\beta_2)N$. So, $\beta_2 \subseteq \beta_1$. Using the symmetry, we obtain that $\beta_1 = \beta_2$. □

Lemma 2.6 *Let $(X, \beta, \mathcal{B})$ be a separated local proximity space. Set $\tau = \tau_{(X,\beta,\mathcal{B})}$. Let $\rho = \beta_{|RC(X,\tau)}$ and $\mathbb{B} = \mathcal{B} \cap RC(X, \tau)$. Then $(RC(X, \tau), \rho, \mathbb{B}) \in \mathcal{L}_{ad}(X, \tau)$.*

Proof. The fact that $(RC(X,\tau),\rho,\mathbb{B})$ is an LCA is proved in [22, Example 40]. The rest can be easily checked. $\square$

Lemma 2.7 *Let (X,τ) be a Tychonoff space and $(RC(X),\rho,\mathbb{B})\in\mathcal{L}_{ad}(X)$. Then there exists a unique separated local proximity space $(X,\beta,\mathcal{B})$ on (X,τ) such that $\mathbb{B}=RC(X)\cap\mathcal{B}$ and $\beta_{|RC(X)}=\rho$. In more details, we set $\mathcal{B}=\{M\subseteq X\mid \exists B\in\mathbb{B}$ such that $M\subseteq B\}$, and for every $M,N\subseteq X$, we put $M(-\beta)N \iff \forall B\in\mathcal{B}\ \exists F,G\in RC(X)$ such that $M\cap B\subseteq \mathrm{int}_X(F),\ N\cap B\subseteq \mathrm{int}_X(G)$ and $F(-\rho)G$.*

Proof. The proof that $(X,\beta,\mathcal{B})$ is a separated local proximity space on (X,τ) is straightforward; for verifying the axiom (BC1) we use Theorem 2.2. The uniqueness follows from Lemma 2.5. $\square$

Definition 2.8 Let X be a locally compact Hausdorff space. We will denote by $\mathcal{L}_a(X)$ the set of all LCAs of the form $(RC(X),\rho,\mathbb{B})$ which satisfy the following conditions:

(LA1) for every $F,G\in RC(X)$, $F\cap G\neq\emptyset$ implies $F\rho G$;

(LA2) $CR(X)\subseteq\mathbb{B}$;

(LA3) for every $F\in RC(X)$ and every $G\in CR(X)$, $F\rho G$ implies $F\cap G\neq\emptyset$.

If $(A,\rho_i,\mathbb{B}_i)\in\mathcal{L}_a(X)$, where $i=1,2$, we set $(A,\rho_1,\mathbb{B}_1)\preceq_l(A,\rho_2,\mathbb{B}_2)$ if $\rho_2\subseteq\rho_1$ and $\mathbb{B}_2\subseteq\mathbb{B}_1$.

Theorem 2.9 *Let (X,τ) be a locally compact Hausdorff space. Then there exists an isomorphism μ between the posets $(\mathcal{L}(X),\leq)$ and $(\mathcal{L}_a(X),\preceq_l)$.*

Proof. It follows immediately from Theorem 2.2. $\square$

Notation 2.10 If $(A,\rho,\mathbb{B})$ is a CLCA then we will write $\rho\subseteq_{\mathbb{B}} C$ provided that C is a normal contact relation on A satisfying the following conditions:

(RC1) $\rho\subseteq C$, and

(RC2) for every $a\in A$ and every $b\in\mathbb{B}$, aCb implies $a\rho b$.

If $\rho\subseteq_{\mathbb{B}} C_1$ and $\rho\subseteq_{\mathbb{B}} C_2$ then we will write $C_1\preceq_c C_2$ iff $C_2\subseteq C_1$.

Corollary 2.11 *Let (X,τ) be a locally compact Hausdorff space and set $(A,\rho,\mathbb{B})=(RC(X),\rho_X,CR(X))$. Then there exists an isomorphism*

$$\mu_c:(\mathcal{K}(X),\leq)\longrightarrow(\mathcal{K}_a(X),\preceq_c),$$

where $\mathcal{K}_a(X)$ is the set of all normal contact relations C on A such that $\rho\subseteq_{\mathbb{B}} C$ (see 2.10 for the notations).

Proof. It follows immediately from Theorem 2.9. $\square$

Proposition 2.12 *Let (X,τ) be a locally compact non-compact Hausdorff space and set $(A,\rho,\mathbb{B}) = (RC(X),\rho_X,CR(X))$. Then C_ρ (see 1.7 for this notation) is the smallest element of the poset $(\mathcal{K}_a(X),\preceq_c)$; hence, if $(\alpha X,\alpha)$ is the Alexandroff (one-point) compactification of X then $\mu_c([(\alpha X,\alpha)]) = C_\rho$ (see Corollary 2.11 for μ_c). Further, the poset $(\mathcal{K}_a(X),\preceq_c)$ has a greatest element $C_{\beta\rho}$; it is defined as follows: for every $a,b\in A$, $a(-C_{\beta\rho})b$ iff there exists a set $\{c_d\in A\mid d\in\mathbb{D}\}$ such that:*
(1) $a\ll_\rho c_d\ll_\rho b^*$, for all $d\in\mathbb{D}$, and
(2) for any two elements d_1,d_2 of $\mathbb{D}$, $d_1<d_2$ implies that $c_{d_1}\ll_\rho c_{d_2}$.
Hence, if $(\beta X,\beta)$ is the Stone-Čech compactification of X then $\mu_c([(\beta X,\beta)]) = C_{\beta\rho}$.

Proof. It is straightforward. □

Remark 2.13 The definition of the relation $C_{\beta\rho}$ in Proposition 2.12 is given in the language of contact relations. It is clear that if we use the fact that all happens in a topological space X then we can define the relation $C_{\beta\rho}$ by setting for every $a,b\in A$, $a(-C_{\beta\rho})b$ iff a and b are completely separated.

Proposition 2.14 *Let X be a locally compact non-compact Hausdorff space. Set $(A,\rho,\mathbb{B}) = (RC(X),\rho_X,CR(X))$ and let $\{C_m\mid m\in M\}$ be a subset of $\mathcal{K}_a(X)$ (see 2.11 for $\mathcal{K}_a(X)$). For every $a,b\in A$, put $a(-C)b$ iff there exists a set $\{c_d\in A\mid d\in\mathbb{D}\}$ such that:*
(1) $a\ll_{C_m} c_d\ll_{C_m} b^*$, for all $d\in\mathbb{D}$ and for each $m\in M$, and
(2) for any two elements d_1,d_2 of $\mathbb{D}$, $d_1<d_2$ implies that $c_{d_1}\ll_{C_m} c_{d_2}$, for every $m\in M$.
Then C is the supremum in $(\mathcal{K}_a(X),\preceq_c)$ of the set $\{C_m\mid m\in M\}$.

Proof. The proof is straightforward. □

3 Extensions over Local Compactifications

Theorem 3.1 *Let, for $i=1,2$, (X_i,τ_i) be a Tychonoff space, (Y_i,f_i) be a Hausdorff local compactification of (X_i,τ_i), $(RC(X_i),\eta_i,\mathbb{B}_i) = \alpha_{X_i}([(Y_i,f_i)])$ (see (5) and (3) for α_{X_i}), and $f: X_1\longrightarrow X_2$ be a continuous function. Then there exists a continuous function $g = L(f): Y_1\longrightarrow Y_2$ such that $g\circ f_1 = f_2\circ f$ iff f satisfies the following conditions:*
(REQ1) *For every $F,G\in RC(X_2)$, $\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F)))\eta_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(G)))$ implies that $F\eta_2 G$;*
(REQ2) *For every $F\in\mathbb{B}_1$ there exists $G\in\mathbb{B}_2$ such that $f(F)\subseteq G$.*

First Proof. Set $\mathcal{B}_i = \{M\subseteq X_i\mid \exists B\in\mathbb{B}_i$ such that $M\subseteq B\}$, where $i=1,2$. For every $M,N\subseteq X_i$, set $M(-\eta_i')N \iff \forall B\in\mathcal{B}_i\ \exists F,G\in RC(X_i)$ such that

$M \cap B \subseteq \mathrm{int}_{X_i}(F)$, $N \cap B \subseteq \mathrm{int}_{X_i}(G)$ and $F(-\eta_i)G$, where $i = 1, 2$. Then, by 2.7, for $i = 1, 2$, $(X_i, \eta'_i, \mathcal{B}_i)$ is the unique separated local proximity space such that $\mathcal{B}_i \cap RC(X_i) = \mathrm{I\!B}_i$ and $(\eta'_i)_{|RC(X_i)} = \eta_i$. So, if we prove that $f : (X_1, \eta'_1, \mathcal{B}_1) \longrightarrow (X_2, \eta'_2, \mathcal{B}_2)$ is equicontinuous iff it satisfies conditions (REQ1) and (REQ2), our assertion will follow from Leader's Theorem 1.19.

It is easy to see that f satisfies condition (EQ2) iff it satisfies condition (REQ2). Let f be an equicontinuous function, $F_1, F_2 \in RC(X_2)$ and

$$\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1)))\eta_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2))).$$

Then $\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1)))\eta'_1\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2)))$ and thus

$$f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_1))))\eta'_2 f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_2)))).$$

Since, for $i = 1, 2$, $f(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F_i))) \subseteq \mathrm{cl}_{X_2} f((\mathrm{int}_{X_1}(f^{-1}(F_i)))) \subseteq F_i$, we get that $F_1\eta'_2F_2$ and, therefore, $F_1\eta_2F_2$. Hence, f satisfies condition (REQ1). So, every equicontinuous function satisfies conditions (REQ1) and (REQ2). Conversely, let f satisfies conditions (REQ1) and (REQ2), $M, N \subseteq X_1$ and $M\eta'_1N$. Then there exists $B \in \mathcal{B}_1$ such that for every $H_1, H_2 \in RC(X_1)$ with $M \cap B \subseteq \mathrm{int}_{X_1}(H_1)$ and $N \cap B \subseteq \mathrm{int}_{X_1}(H_2)$, $H_1\eta_1H_2$ holds. Suppose that $f(M)(-\eta'_2)f(N)$. Then, for every $C \in \mathcal{B}_2$ there exist $F, G \in RC(X_2)$ such that $f(M) \cap C \subseteq \mathrm{int}_{X_2}(F)$, $f(N) \cap C \subseteq \mathrm{int}_{X_2}(G)$ and $F(-\eta_2)G$. Since condition (REQ2) implies condition (EQ2), we have that $f(B) \in \mathcal{B}_2$. Thus there exist $F, G \in RC(X_2)$ such that $f(M) \cap f(B) \subseteq \mathrm{int}_{X_2}(F)$, $f(N) \cap f(B) \subseteq \mathrm{int}_{X_2}(G)$ and $F(-\eta_2)G$. Then $M \cap B \subseteq \mathrm{int}_{X_1}(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(F))))$ and $N \cap B \subseteq \mathrm{int}_{X_1}(\mathrm{cl}_{X_1}(\mathrm{int}_{X_1}(f^{-1}(G))))$. Hence, by (REQ1), $F\eta_2G$ holds, a contradiction. Therefore, $f(M)\eta'_2f(N)$. Thus, f is an equicontinuous function.

Second Proof. In the first proof we used the Leader Local Compactification Theorem 1.19. We will now give another proof which does not use Leader's theorem. Hence, by the First Proof, it will imply the second part of Leader Theorem 1.19. The more important thing is that the method of this new proof will be used later on for the proof of our Main Theorem 3.5.

($\Rightarrow$) Let there exists a continuous function $g : Y_1 \longrightarrow Y_2$ such that $g \circ f_1 = f_2 \circ f$. Then, using the notations of (4), we have, by the proof of Theorem 2.2, that the maps $r_i = r_{(Y_i, f_i)}$ are LCA-isomorphisms, $i = 1, 2$. Set, for $i = 1, 2$, $e_i = (r_i)^{-1}$ and $\rho_i = \rho_{Y_i}$. Then, by 1.20, for every $F \in RC(X_i)$ and $i = 1, 2$, $e_i(F) = \mathrm{cl}_{Y_i}(f_i(F))$. Let $\varphi_g = \Lambda^t(g)$ (see Theorem 1.16), i.e.

(8) $\varphi_g : (RC(Y_2), \rho_2, CR(Y_2)) \longrightarrow (RC(Y_1), \rho_2, CR(Y_1)),\ G \mapsto \mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(G)))$.

Set also

(9) $\varphi_f = r_1 \circ \varphi_g \circ e_2 : (RC(X_2), \eta_2, \mathrm{I\!B}_2) \longrightarrow (RC(X_1), \eta_1, \mathrm{I\!B}_1)$.

We will prove that

(10) $\varphi_f(G) = \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$, for every $G \in RC(X_2)$.

Indeed, let $G \in RC(X_2)$. Then $\varphi_f(G) = (f_1)^{-1}(\mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))))) = \mathrm{cl}_{X_1}((f_1)^{-1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G))))))$. It is easy to see that

$$(f_2)^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))) = \mathrm{int}_{X_2}(G).$$

Thus we obtain that $(f_1)^{-1}(g^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G))))) = \{x \in X_1 \mid (g \circ f_1)(x) \in \mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))\} = \{x \in X_1 \mid f_2(f(x)) \in \mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G)))\} = \{x \in X_1 \mid f(x) \in (f_2)^{-1}(\mathrm{int}_{Y_2}(\mathrm{cl}_{Y_2}(f_2(G))))\} = \{x \in X_1 \mid f(x) \in \mathrm{int}_{X_2}(G)\} = f^{-1}(\mathrm{int}_{X_2}(G))$. Now it becomes clear that (10) holds.

Since, by Theorem 1.16, φ_g is a **DHLC**-morphism, we get that φ_f is a **DHLC**-morphism. Therefore, by (DLC4), for every $F \in \mathbb{B}_1$ there exists $G \in \mathbb{B}_2$ such that $F \subseteq \varphi_f(G)$. Since g is continuous, we get that f is continuous. Thus $f(F) \subseteq f(\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G))) \subseteq \mathrm{cl}_{X_2}(f(f^{-1}(\mathrm{int}_{X_2}(G)))) \subseteq G$. Hence, condition (REQ2) is checked. Let now $F, G \in RC(X_1)$ and $F \ll_{\eta_1} G$. Then, by condition (DLC3S), $(\varphi_f(F^*))^* \ll_{\eta_2} \varphi_f(G)$. It is easy to see now that condition (REQ1) is also fulfilled.
($\Leftarrow$) Let f be a function satisfying conditions (REQ1) and (REQ2). Set $\varphi_f : (RC(X_2), \eta_2, \mathbb{B}_2) \longrightarrow (RC(X_1), \eta_1, \mathbb{B}_1)$, $G \mapsto \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$. Then it is easy to check that φ_f is a **DHLC**-morphism. Put $g = \Lambda^a(\varphi_f)$. Then g is a continuous function and $g : \Lambda^a(RC(X_1), \eta_1, \mathbb{B}_1) \longrightarrow \Lambda^a(RC(X_2), \eta_2, \mathbb{B}_2)$, i.e. $g : Y_1 \longrightarrow Y_2$ (see the prof of Theorem 2.2). We will show that $g \circ f_1 = f_2 \circ f$. Let $x \in X_1$. Then $g(f_1(x)) = g(\sigma_x)$ and $f_2(f(x)) = \sigma_{f(x)}$. By Theorem 1.16, we have that $g(\sigma_x) \cap \mathbb{B}_2 = \{G \in \mathbb{B}_2 \mid \forall F \in RC(X_2), (G \ll_{\eta_2} F) \rightarrow (x \in \varphi_f(F))\}$. We will prove that $\{G \in \mathbb{B}_2 \mid \forall F \in RC(X_2), (G \ll_{\eta_2} F) \rightarrow (x \in \varphi_f(F))\} = \{G \in \mathbb{B}_2 \mid f(x) \in G\}$. This will imply, by 1.14, the desired equality. So, let $G \in \mathbb{B}_2$ and $f(x) \in G$. Let $F \in RC(X_2)$ and $G \ll_{\eta_2} F$. Using condition (A1), we get that $G \ll_{\rho_{X_2}} F$, i.e. that $G \subseteq \mathrm{int}_{X_2}(F)$. Thus we obtain that $x \in f^{-1}(G) \subseteq f^{-1}(\mathrm{int}_{X_2}(F) \subseteq \varphi_f(F)$. Conversely, let $G \in \mathbb{B}_2 \cap g(\sigma_x)$. Suppose that $f(x) \notin G$. Then $f(x) \in X_2 \backslash G = \mathrm{int}_{X_2}(G^*)$. By condition (A2), there exists $F \in \mathbb{B}_2$ such that $f(x) \in \mathrm{int}_{X_2}(F)$ and $F \ll_{\rho_2} G^*$. Then $G \ll_{\rho_2} F^*$. Hence $x \in \varphi_f(F^*)$. Since $f(x) \in \mathrm{int}_{X_2}(F) = X_2 \setminus F^*$, we get a contradiction. Therefore, $f(x) \in G$. Thus, $g \circ f_1 = f_2 \circ f$. $\square$

It is natural to write $f : (X_1, RC(X_1), \rho_1, \mathbb{B}_1) \longrightarrow (X_2, RC(X_2), \rho_2, \mathbb{B}_2)$ when we have a situation like that which is described in Theorem 3.1. Then, in analogy with the Leader's equicontinuous functions (see Leader's Theorem 1.19), the continuous functions $f : (X_1, RC(X_1), \rho_1, \mathbb{B}_1) \longrightarrow (X_2, RC(X_2), \rho_2, \mathbb{B}_2)$ which satisfy conditions (REQ1) and (REQ2) will be called *R-equicontinuous functions.*

Recall that a function $f : X \longrightarrow Y$ is called *skeletal* ([16]) if

$$\mathrm{int}(f^{-1}(\mathrm{cl}(V))) \subseteq \mathrm{cl}(f^{-1}(V)) \tag{11}$$

for every open subset V of Y. Recall also the following three results:

Lemma 3.2 ([4]) *Let $f : X \longrightarrow Y$ be a continuous map. Then the following conditions are equivalent:*
(a) f is a skeletal map;
(b) For every $F \in RC(X)$, $\mathrm{cl}(f(F)) \in RC(Y)$.

Lemma 3.3 ([7]) *A continuous map $f : X \longrightarrow Y$, where X and Y are topological spaces, is skeletal iff for every open dense subset V of Y, $\mathrm{cl}_X(f^{-1}(V)) = X$ holds.*

Lemma 3.4 ([7]) *Let (X_i, τ_i), $i = 1, 2$, be two topological spaces, (Y_i, f_i) be some extensions of (X_i, τ_i), $i = 1, 2$, $f : X_1 \longrightarrow X_2$ and $g : Y_1 \longrightarrow Y_2$ be two continuous functions such that $g \circ f_1 = f_2 \circ f$. Then g is skeletal iff f is skeletal.*

We are now ready to prove the main result of this paper:

Theorem 3.5 *Let, for $i = 1, 2$, (X_i, τ_i) be a Tychonoff space, (Y_i, f_i) be a Hausdorff local compactification of (X_i, τ_i), $(RC(X_i), \eta_i, \mathbb{B}_i) = \alpha_{X_i}([(Y_i, f_i)])$ (see (5) and (3) for α_{X_i}), $f : (X_1, RC(X_1), \rho_1, \mathbb{B}_1) \longrightarrow (X_2, RC(X_2), \rho_2, \mathbb{B}_2)$ be an R-equicontinuous function and $g = L(f) : Y_1 \longrightarrow Y_2$ be the continuous function such that $g \circ f_1 = f_2 \circ f$ (its existence is guaranteed by Theorem 3.1). Then:*

(a) g is skeletal iff f is skeletal;

(b) g is an open map iff f is a skeletal map and satisfies the following condition:

(O) $\forall F \in \mathbb{B}_1$ *and* $\forall G \in RC(X_1), (F \ll_{\rho_1} G) \rightarrow (\mathrm{cl}_{X_2}(f(F)) \ll_{\rho_2} \mathrm{cl}_{X_2}(f(G)))$;

(b′) g is an open map iff f satisfies the following condition:

(O1) $\forall F \in \mathbb{B}_1$ *and* $\forall G \in RC(X_1), (F \ll_{\rho_1} G) \rightarrow (\mathrm{cl}_{X_2}(f(F)) \ll_{\rho'_2} \mathrm{cl}_{X_2}(f(G)))$, *where ρ'_2 is the unique separated local proximity on X_2 such that $(\rho'_2)_{|RC(X_2)} = \rho_2$ (see 2.7);*

(b″) g is an open map iff f satisfies the following condition:

(O2) $\forall A \subseteq X_1$ *such that there exists $F \in \mathbb{B}_1$ with $A \subseteq F$, and $\forall B \subseteq X_1$, $(A \ll_{\rho'_1} B) \rightarrow (f(A) \ll_{\rho'_2} \mathrm{cl}_{X_2}(f(B)))$, where, for $i = 1, 2$, ρ'_i is the unique separated local proximity on X_i such that $(\rho'_i)_{|RC(X_i)} = \rho_i$ (see 2.7);*

(c) g is a perfect map iff f satisfies the following condition:

(P) *For every $G \in \mathbb{B}_2$, $\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G))) \in \mathbb{B}_1$ holds;*

(d) $\mathrm{cl}_{Y_2}(g(Y_1)) = Y_2$ iff $\mathrm{cl}_{X_2}(f(X_1)) = X_2$;

(e) g is an injection iff f satisfies the following condition:

(I) *For every $F_1, F_2 \in \mathbb{B}_1$ such that $F_1(-\rho_1)F_2$ there exist $G_1, G_2 \in \mathbb{B}_2$ with $G_1 \ll_{\rho_2} G_2$, $F_1 \subseteq \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G_2)))$ and $\mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G_2)))(-\rho_1)F_2$;*

(f) g is an open injection iff f satisfies condition (O1) *(or, equivalently, f is skeletal and satisfies condition (O)) and the following one:*

(OI) $\forall F \in RC(X_1)\ \exists G \in RC(X_2)$ *such that* $F = \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(G)))$;

(g) g is a perfect surjection iff f satisfies condition (P) *and* $\mathrm{cl}_{X_2}(f(X_1)) = X_2$.

Proof. Set $\varphi_g = \Lambda^t(g)$ (see Theorem 1.16). Then $\varphi_g : RC(Y_2) \longrightarrow RC(Y_1)$, $G \mapsto \mathrm{cl}_{Y_1}(g^{-1}(\mathrm{int}_{Y_2}(G)))$. Set also $\varphi_f : RC(X_2) \longrightarrow RC(X_1)$, $F \mapsto \mathrm{cl}_{X_1}(f^{-1}(\mathrm{int}_{X_2}(F)))$. Then, (8), (9) and (10) imply that $\varphi_f = r_1 \circ \varphi_g \circ e_2$.

(a) It follows from Lemma 3.4.

(b) Since every open map is skeletal, we get, using (a), that if g is an open map then f is skeletal. So, we can suppose that f is skeletal. Then, as it follows from the proof of [4, Theorem 2.11], φ_f is a complete Boolean homomorphism. Thus, by [4, (33)], the map φ_f has a left adjoint $\varphi^f : RC(X_1) \longrightarrow RC(X_2)$, $F \mapsto \mathrm{cl}_{X_2}(f(F))$. Further, by the proof of Theorem 2.2, the maps $r_1 : (RC(Y_1), \rho_{Y_1}, CR(Y_1)) \longrightarrow (RC(X_1), \rho_1, \mathbb{B}_1)$, $G \mapsto f_1^{-1}(G)$ and $e_2 = r_2^{-1}$ are LCA-isomorphisms. Hence, (9) and 1.18 imply that g is an open map iff the map f (is skeletal and) satisfies the following condition:

(O$'$) $\forall F \in \mathbb{B}_1$ and $\forall G \in RC(X_2), (\varphi^f(F)\rho_2 G) \rightarrow (F\rho_1\varphi_f(G))$.

It is easy to see that condition (O$'$) is equivalent to the following one:

(O$''$) $\forall F \in \mathbb{B}_1$ and $\forall G \in RC(X_2), (F \ll_{\rho_1} \varphi_f(G)) \rightarrow (\varphi^f(F) \ll_{\rho_2} G)$.

We will prove that condition (O$''$) is equivalent to condition (O). Indeed, let $F \in \mathbb{B}_1$, $G \in RC(X_1)$ and $F \ll_{\rho_1} G$. Since f is skeletal, the map φ^f exists. Set $H = \varphi^f(G)$. Thus, $H \in RC(X_2)$ and $\varphi_f(H) = \varphi_f(\varphi^f(G)) \supseteq G$. Therefore $F \ll_{\rho_1} \varphi_f(H)$. Then (O$''$) implies that $\varphi^f(F) \ll_{\rho_2} H$, i.e. $\varphi^f(F) \ll_{\rho_2} \varphi^f(G)$. So, condition (O) is satisfied.

Conversely, let f satisfies condition (O), $F \in \mathbb{B}_1$, $G \in RC(X_2)$ and $F \ll_{\rho_1} \varphi_f(G)$. Then, by (O), $\varphi^f(F) \ll_{\rho_2} \varphi^f(\varphi_f(G))$. Since $\varphi^f(\varphi_f(G)) \subseteq G$, we get that $\varphi^f(F) \ll_{\rho_2} G$. Thus, condition (O$''$) is fulfilled.

(b$'$) Having in mind Lemma 3.2, we need only to show that if f satisfies condition (O1) then f is a skeletal map. So, let f satisfies condition (O1), V be an open dense subset of X_2 and $G = \mathrm{cl}_{X_1}(f^{-1}(V))$. Then $G \in RC(X_1)$. Suppose that $G \neq X_1$. Then there exists $x \in X_1 \setminus G$. Clearly, $f_1(x) \notin \mathrm{cl}_{Y_1}(f_1(G))$. Since Y_1 is locally compact and Hausdorff, we get that there exists $F \in \mathbb{B}_1$ such that $x \in F$ and $F(-\rho_1)G$. Thus $F \ll_{\rho_1} G^*$. Therefore, by (O1), $\mathrm{cl}_{X_2}(f(F)) \ll_{\rho_2'} \mathrm{cl}_{X_2}(f(G^*))$. Set $U = X_1 \setminus G$. Since f is continuous, we have that $H = \mathrm{cl}_{X_2}(f(G^*)) = \mathrm{cl}_{X_2}(f(U)) \subseteq \mathrm{cl}_{X_2}(f(X_1 \setminus f^{-1}(V))) = \mathrm{cl}_{X_2}(f(X_1) \setminus V) \subseteq X_2 \setminus V$. Thus $H' = \mathrm{cl}_{X_2}(X_2 \setminus H) \supseteq \mathrm{cl}_{X_2}(V) = X_2$. We get that $\mathrm{cl}_{X_2}(f(F))(-\rho_2')X_2$, a contradiction. Thus, $f^{-1}(V)$ is dense in X_1. Then Lemma 3.3 implies that f is a skeletal map.

(b$''$) It is enough to show that conditions (O1) and (O2) are equivalent. Set $\mathbb{B}_1' = \{A \subseteq X_1 \mid \exists F \in \mathbb{B}_1 \text{ such that } A \subseteq F\}$.

Let f satisfies condition (O1), $A \in \mathbb{B}_1', B \subseteq X_1$ and $A \ll_{\rho_1'} B$. Then $A(-\rho_1')(X_1 \setminus B)$. Thus $\mathrm{cl}_{Y_1}(f_1(A)) \cap \mathrm{cl}_{Y_1}(f_1(X_1 \setminus B)) = \emptyset$. Since $A \in \mathbb{B}_1'$, we have that $\mathrm{cl}_{Y_1}(f_1(A))$ is a compact subset of Y_1. Using the fact that Y_1 is a locally compact Hausdorff space, we get that there exist $F \in RC(X_1)$ and $U \in RO(X_1)$ such that $A \subseteq F$, $X_1 \setminus B \subseteq U$ and $F(-\rho_1')U$. Set $G = X_1 \setminus U$. Then $G \in RC(X_1)$ and $F \ll_{\rho_1} G$. Thus, by (O1), $\mathrm{cl}_{X_2}(f(F)) \ll_{\rho_2'} \mathrm{cl}_{X_2}(f(G))$. Since $G \subseteq B$, we get that $f(A) \ll_{\rho_2'} \mathrm{cl}_{X_2}(f(B))$. So, f satisfies condition (O2). Obviously, condition (O2) implies condition (O1). Therefore, conditions (O1) and (O2) are equivalent.

(c) By [11, Theorem 3.7.18], g is a perfect map iff φ_g satisfies the following condition: for every $G \in CR(Y_2)$, $\varphi_g(G) \in CR(Y_1)$ holds. Having in mind the proof of Theorem 2.2 and (9), we get that g is a perfect map iff f satisfies condition (P).

(d) This is obvious.

(e) Using again (9), our assertion follows from [6, Theorem 3.16].

(f) It follows from (b), (9), and [6, Theorem 3.23].

(g) It follows from (c) and (d). $\square$

Recall that a continuous map $f : X \longrightarrow Y$ is called *quasi-open* ([15]) if for every non-empty open subset U of X, $\mathrm{int}(f(U)) \neq \emptyset$ holds. As it is shown in [4], if X is regular and Hausdorff, and $f : X \longrightarrow Y$ is a closed map, then f is quasi-open iff f is skeletal. This fact and Theorem 3.5 imply the following two corollaries:

Corollary 3.6 *Let (X_1, δ_1), (X_2, δ_2) be two Efremovič proximity spaces, $(cX_i, c_i) = L(X_i, \delta_i)$ (see 1.19 for this notation) be the Hausdorff compactification of (X_i, τ_{δ_i}) corresponding, by the Smirnov Compactification Theorem [21], to the Efremovič proximity space (X_i, δ_i), where $i = 1, 2$, $f : (X_1, \delta_1) \longrightarrow (X_2, \delta_2)$ be a proximally continuous function, and $g = L(f) : cX_1 \longrightarrow cX_2$ be the continuous function such that $g \circ c_1 = c_2 \circ f$ (see 1.19 for its existence). Then:*

(a) g is quasi-open iff f is skeletal;

(b)(V. Z. Poljakov [18]) *g is an open map iff f satisfies the following condition:*

(OC) *For every $A, B \subseteq X_1$ such that $A \ll_{\delta_1} B$, $f(A) \ll_{\delta_2} \mathrm{cl}_{X_2}(f(B))$ holds.*

Corollary 3.7 *Let X_1, X_2 be two Tychonoff spaces, $f : X_1 \longrightarrow X_2$ be a continuous function and $\beta f : \beta X_1 \longrightarrow \beta X_2$ be the extension of f to the Stone-Čech compactifications of X_1 and X_2. Then:*

(a) βf is quasi-open iff f is skeletal;

(b) βf is an open map iff f satisfies the following condition:

(OB) *For every $A, B \subseteq X_1$ which are completely separated in X_1, $f(A)$ and $X_2 \setminus \mathrm{cl}_{X_2}(f(X_1 \setminus B))$ are completely separated in X_2;*

(c)(V. Z. Poljakov [18]) *If X_1 and X_2 are normal spaces then βf is open iff for every $A, B \subseteq X_1$ such that $\mathrm{cl}_{X_1}(A) \subseteq \mathrm{int}_{X_1}(B)$, $\mathrm{cl}_{X_2}(f(A)) \subseteq \mathrm{int}_{X_2}(\mathrm{cl}_{X_2}(f(B)))$ holds;*

(d)(A. D. Taĭmanov [1]) *If X_1 and X_2 are normal spaces and f is an open and closed map then βf is open.*

Remark 3.8 In [18], after establishing the general result 3.6(b), V. Z. Poljakov writes (in the notations of Corollary 3.7) that βf is open iff for every two completely separated subsets A and B of X_1, the sets $f(A)$ and $\{y \in X_2 \mid f^{-1}(y) \subseteq B\}$ are completely separated in X_2. Since $\{y \in X_2 \mid f^{-1}(y) \subseteq B\} = f^\sharp(B) = X_2 \setminus f(X_1 \setminus B)$, we get that Poljakov's condition implies condition (OB) and thus it is sufficient for the openness of βf. It is, however, not necessary. Indeed, let $f : \mathbb{Q} \longrightarrow \beta\mathbb{Q}$ be the inclusion map. Then $\beta f : \beta\mathbb{Q} \longrightarrow \beta\mathbb{Q}$ is the identity map and hence it is an open map. Let $A, B \subseteq \mathbb{Q}$ and A, B be completely separated in $\mathbb{Q}$. Then, by Poljakov's condition, the sets $f(A)$ and $f^\sharp(B)$ are completely separated in $\beta\mathbb{Q}$, i.e. $\mathrm{cl}_{\beta\mathbb{Q}}(f(A)) \cap \mathrm{cl}_{\beta\mathbb{Q}}(f^\sharp(B)) = \emptyset$. Since $f^\sharp(B) = f(B) \cup (\beta\mathbb{Q} \setminus \mathbb{Q})$, we get that $\mathrm{cl}_{\beta\mathbb{Q}}(f^\sharp(B)) = \beta\mathbb{Q}$. Thus $f(A)$ and $\beta\mathbb{Q}$ are completely separated in $\beta\mathbb{Q}$, a contradiction. Hence, the map f does not satisfy Poljakov's condition.

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